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Nonlinear superposition formulae of the Ito equation and a model equation for shallow water waves

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Abstract. In this paper, nonlinear superposition formulae of the Ito equation and a model equation for shallow water waves are proved under certain conditions. Some particular solutions of the Ito equation are obtained as an illustrative application of the obtained nonlinear superposition formula.

1. Introduction

As is well known, it is difficult to find particular solutions of a nonlinear differential equation in mathematics and physics. It is also difficult to find out another solution from a given solution of such an equation. However, recent research suggests that a new solution can usually be obtained from a given solution of an equation if the so-called Backlund transformation (BT) for the equation is found. Furthermore, if a nonlinear superposition formula of a BT for a nonlinear differential equation is derived, we can transform the problem of seeking exact solutions of the equation into purely algebraic operations. In general, we can derive a nonlinear superposition formula from the commutability of the BT. Unfortunately, a rigorous proof of the commutability of the BT for a general nonlinear evolution equation is lacking [1, 2]. Therefore, it is necessary to prove a nonlinear superposition formula directly. Until now, some progress has been made in this field. To our knowledge, the main results on bilinear operator equations are as follows. In 1978, Hirota and Satsuma obtained nonlinear superposition formulae of KdV, MKdV, SG, etc, in bilinear form [3]. Since 1979, Nakamura has proved nonlinear superposition formulae of BO [4], cKdV [5], MMKdV [6] and KP [7], respectively. During recent years, we have proved nonlinear superposition formulae of a model equation for shallow water waves [8], a fifth- and seventh-order KdV equation [9, 10], DJKM equation [11], KdV, MKdV, MMKdV hierarchies [12] and Boussinesq hierarchies [13]. It is worth noticing that the nonlinear superposition formulae of the above-mentioned equations possess a unified and simple structure,

$$f_0 f_{12} = k(D_x + \lambda_1 - \lambda_2) f_1 \cdot f_2 \tag{1}$$

where λ_1 and λ_2 are parameters related to solutions f_0, f_1, f_2 and the BT, and the bilinear operator $D_x^m D_t^n$ is defined by [14]

$$D_x^m D_t^n a \cdot b \equiv \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x, t) b(x', t')|_{x'=x, t'=t}.$$

In this paper, we consider the Ito equation and a model equation for shallow water waves and obtain a nonlinear superposition formula which is different from (1). The

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contents are arranged as follows. In section 2, a nonlinear superposition formula of the Ito equation is proved under certain conditions. By using the nonlinear superposition formula, some particular solutions of the Ito equation are obtained in section 3. In section 4, we similarly present a nonlinear superposition formula of a model equation for shallow water waves. Finally, we list some bilinear operator identities in the appendix which are used in this paper.

2. Nonlinear superposition formula of the Ito equation

The Ito equation reads [15]

$$D_t(D_t+D_x^3)f\cdot f=0. \tag{2}$$

Through the dependent variable transform $u=2(\ln f)_{xx}$, (2) becomes

$$u_{tt}+u_{xxxt}+3(2u_xu_t+uu_{xt})+3u_{xx}\int_{-\infty}^{x}u_t\,\mathrm{d}x'=0.$$

Concerning (2), some work has been done. Hirota gave a two-parameter BT for the Ito (2) [15]. We generalized Hirota's result and presented the following BT with three parameters [16]:

$$(D_t+3\gamma^2D_x-3\gamma D_x^2+D_x^3-\lambda)f\cdot f'=0 \tag{3a}$$

$$(D_xD_t-\mu D_x-\gamma D_t+\gamma\mu)f\cdot f'=0 \tag{3b}$$

where λ, μ and γ are arbitrary constants. Jimbo and Miwa have pointed out that the Ito equation is related to Kac–Moody algebra $D_3^{(2)}$ [17]. In this section, we are going to give a nonlinear superposition formula of the Ito equation. In the following discussion, we set $\gamma=\lambda=0$ in BT (3) for the sake of convenience. In this case, (3) becomes

$$(D_t+D_x^3)f\cdot f'=0 \tag{3a'}$$

$$(D_xD_t-\mu D_x)f\cdot f'=0. \tag{3b'}$$

Let f_0 be a solution of the Ito equation (2), $f_0\neq 0$. Suppose that f_i $(i=1,2)$ is a solution of (2) which is related by f_0 under BT (3′) to μ_i, i.e. $f_0\xrightarrow{\mu_i}f_i$ $(i=1,2)$, and that f_{12} is defined by

$$D_xf_0\cdot f_{12}=kD_xf_1\cdot f_2 \qquad \text{(where } k \text{ is a non-zero constant).} \tag{4}$$

From these assumptions, we deduce that

$$\begin{aligned}
0&=[(D_xD_t-\mu_1D_x)f_0\cdot f_1]f_2-[(D_xD_t-\mu_2D_x)f_0\cdot f_2]f_1\\
&\overset{(A1,\,A2)}{=}-f_{0_x}D_tf_1\cdot f_2-f_{0_t}D_xf_1\cdot f_2+\tfrac12 f_0[(D_xf_1\cdot f_2)_t+(D_tf_1\cdot f_2)_x]\\
&\qquad-(\mu_1-\mu_2)f_{0_x}f_1f_2+\tfrac12 f_0[(\mu_1+\mu_2)D_xf_1\cdot f_2+(\mu_1-\mu_2)(f_1f_2)_x]\\
&\overset{(4)}{=}f_{0_x}[-D_tf_1\cdot f_2-(\mu_1-\mu_2)f_1f_2]-\frac1k f_{0_t}D_xf_0\cdot f_{12}+\frac{1}{2k}f_0(D_xf_0\cdot f_{12})_t\\
&\qquad+\frac{1}{2k}(\mu_1+\mu_2)f_0D_xf_0\cdot f_{12}+\tfrac12 f_0[D_tf_1\cdot f_2+(\mu_1-\mu_2)f_1f_2]_x\\
&=f_{0_x}\left(-D_tf_1\cdot f_2-(\mu_1-\mu_2)f_1f_2-\frac1k D_tf_0\cdot f_{12}+\frac1k(\mu_1+\mu_2)f_0f_{12}\right)\\
&\qquad+\tfrac12 f_0\left(D_tf_1\cdot f_2+(\mu_1-\mu_2)f_1f_2+\frac1k(D_tf_0\cdot f_{12})-\frac1k(\mu_1+\mu_2)f_0f_{12}\right)_x
\end{aligned}$$

which implies that

$$D_t f_1 \cdot f_2 + (\mu_1 - \mu_2) f_1 f_2 + \frac{1}{k} D_t f_0 \cdot f_{12} - \frac{1}{k}(\mu_1 + \mu_2) f_0 f_{12} = c_1(t) f_0^2 \tag{5}$$

where $c_1(t)$ is some function of t.

Next, we have

$$\begin{aligned}
0 &= [(D_t + D_x^3) f_0 \cdot f_1] f_2 - [(D_t + D_x^3) f_0 \cdot f_2] f_1 \\
&\overset{(A3,\,A4)}{=} -f_0 D_t f_1 \cdot f_2 - 3 f_{0_{xx}} D_x f_1 \cdot f_2 + 3 f_{0_x} (D_x f_1 \cdot f_2)_x \\
&\quad - \tfrac{1}{4} f_0 [D_x^3 f_1 \cdot f_2 + 3 (D_x f_1 \cdot f_2)_{xx}] \\
&\overset{(4)}{=} -f_0 (D_t f_1 \cdot f_2 + \tfrac{1}{4} D_x^3 f_1 \cdot f_2) - \frac{3}{k} f_{0_{xx}} D_x f_0 \cdot f_{12} \\
&\quad + \frac{3}{k} f_{0_x} (D_x f_0 \cdot f_{12})_x - \frac{3}{4k} f_0 (D_x f_0 \cdot f_{12})_{xx} \\
&= -f_0 \left(D_t f_1 \cdot f_2 + \tfrac{1}{4} D_x^3 f_1 \cdot f_2 + \frac{3}{4k} D_x^3 f_0 \cdot f_{12} \right)
\end{aligned}$$

which implies that

$$D_t f_1 \cdot f_2 + \tfrac{1}{4} D_x^3 f_1 \cdot f_2 + \frac{3}{4k} D_x^3 f_0 \cdot f_{12} = 0 \tag{6}$$

and

$$\begin{aligned}
0 &= [(D_t + D_x^3) f_0 \cdot f_1]_x f_2 - [(D_t + D_x^3) f_0 \cdot f_2]_x f_1 \\
&\overset{(A5,\,A6)}{=} f_{0_t} D_x f_1 \cdot f_2 - f_{0_x} D_t f_1 \cdot f_2 - \tfrac{1}{2} f_0 [(D_x f_1 \cdot f_2)_t + (D_t f_1 \cdot f_2)_x] \\
&\quad - 2 f_{0_{xxx}} D_x f_1 \cdot f_2 + \tfrac{1}{2} f_{0_x} [D_x^3 f_1 \cdot f_2 + 3 (D_x f_1 \cdot f_2)_{xx}] \\
&\quad - \tfrac{1}{2} f_0 [(D_x^3 f_1 \cdot f_2)_x + (D_x f_1 \cdot f_2)_{xxx}] \\
&\overset{(4)}{=} f_{0_x} (-D_t f_1 \cdot f_2 + \tfrac{1}{2} D_x^3 f_1 \cdot f_2) - \tfrac{1}{2} f_0 (D_x^3 f_1 \cdot f_2 + D_t f_1 \cdot f_2)_x \\
&\quad + \frac{1}{k} f_{0_t} D_x f_0 \cdot f_{12} - \frac{1}{2k} f_0 (D_x f_0 \cdot f_{12})_t - \frac{2}{k} f_{0_{xxx}} D_x f_0 \cdot f_{12} \\
&\quad + \frac{3}{2k} f_{0_x} (D_x f_0 \cdot f_{12})_{xx} - \frac{1}{2k} f_0 (D_x f_0 \cdot f_{12})_{xxx} \\
&= f_{0_x} \left(-D_t f_1 \cdot f_2 + \frac{1}{k} D_t f_0 \cdot f_{12} + \tfrac{1}{2} D_x^3 f_1 \cdot f_2 - \frac{1}{2k} D_x^3 f_0 \cdot f_{12} \right) \\
&\quad + f_0 \left(-\tfrac{1}{2} D_t f_1 \cdot f_2 - \frac{1}{2k} D_t f_0 \cdot f_{12} - \tfrac{1}{2} D_x^3 f_1 \cdot f_2 - \frac{1}{2k} D_x^3 f_0 \cdot f_{12} \right)_x \\
&\overset{(6)}{=} f_{0_x} \left(\frac{1}{k} D_t f_0 \cdot f_{12} + \frac{1}{4k} D_x^3 f_0 \cdot f_{12} + \tfrac{3}{4} D_x^3 f_1 \cdot f_2 \right) \\
&\quad - \tfrac{1}{2} f_0 \left(\frac{1}{k} D_t f_0 \cdot f_{12} + \frac{1}{4k} D_x^3 f_0 \cdot f_{12} + \tfrac{3}{4} D_x^3 f_1 \cdot f_2 \right)_x
\end{aligned}$$

which implies that

$$D_t f_0\cdot f_{12}+\tfrac{1}{4}D_x^3 f_0\cdot f_{12}+\frac{3k}{4}D_x^3 f_1\cdot f_2=c_2(t)f_0^2 \tag{7}$$

where $c_2(t)$ is some function of t. Here and in the following, we assume that there exists a f_{12} determined by (4) such that $c_1(t)=0$ in (5) and $c_2(t)=0$ in (7), i.e.

$$D_t f_1\cdot f_2+(\mu_1-\mu_2)f_1 f_2+\frac{1}{k}D_t f_0\cdot f_{12}-\frac{1}{k}(\mu_1+\mu_2)f_0 f_{12}=0 \tag{5'}$$

$$D_t f_0\cdot f_{12}+\tfrac{1}{4}D_x^3 f_0\cdot f_{12}+\frac{3k}{4}D_x^3 f_1\cdot f_2=0. \tag{7'}$$

Now we can show that the corresponding f_{12} is a new solution of (2) which is related to f_1 and f_2 under BT (3′). In fact, we have

$$\begin{aligned}
-Q_1 f_0 &\equiv -[(D_xD_t-\mu_2 D_x)f_1\cdot f_{12}]f_0\\
&=[(D_xD_t+\mu_1 D_x)f_1\cdot f_0]f_{12}-[(D_xD_t-\mu_2 D_x)f_1\cdot f_{12}]f_0\\
&\overset{(A1,\,A2)}{=}-f_{1_x}D_t f_0\cdot f_{12}-f_{1_t}D_x f_0\cdot f_{12}+\tfrac{1}{2}f_1[(D_x f_0\cdot f_{12})_t+(D_t f_0\cdot f_{12})_x]\\
&\quad+(\mu_1+\mu_2)f_{1_x}f_0 f_{12}-\tfrac{1}{2}f_1[(\mu_1-\mu_2)D_x f_0\cdot f_{12}+(\mu_1+\mu_2)(f_0 f_{12})_x]\\
&\overset{(4)}{=}f_{1_x}[-D_t f_0\cdot f_{12}+(\mu_1+\mu_2)f_0 f_{12}]-kf_{1_t}D_x f_1\cdot f_2+\frac{k}{2}f_1(D_x f_1\cdot f_2)_t\\
&\quad-\frac{k}{2}(\mu_1-\mu_2)f_1 D_x f_1\cdot f_2+\tfrac{1}{2}f_1[D_t f_0\cdot f_{12}-(\mu_1+\mu_2)f_0 f_{12}]_x\\
&=f_{1_x}[-D_t f_0\cdot f_{12}+(\mu_1+\mu_2)f_0 f_{12}-kD_t f_1\cdot f_2-k(\mu_1-\mu_2)f_1 f_2]\\
&\quad+\tfrac{1}{2}f_1[D_t f_0\cdot f_{12}-(\mu_1+\mu_2)f_0 f_{12}+kD_t f_1\cdot f_2+k(\mu_1-\mu_2)f_1 f_2]_x\\
&\overset{(5')}{=}0.
\end{aligned}$$

Thus $Q_1=0$, i.e $(D_xD_t-\mu_2 D_x)f_1\cdot f_{12}=0$. Similarly, we can show that $(D_xD_t-\mu_1 D_x)f_2\cdot f_{12}=0$. Besides, we have

$$\begin{aligned}
-Q_2 f_0 &\equiv -[(D_t+D_x^3)f_1\cdot f_{12}]f_0\\
&=[(D_t+D_x^3)f_1\cdot f_0]f_{12}-[(D_t+D_x^3)f_1\cdot f_{12}]f_0\\
&\overset{(A3,\,A4)}{=}-f_1 D_t f_0\cdot f_{12}-3f_{1_{xx}}D_x f_0\cdot f_{12}+3f_{1_x}(D_x f_0\cdot f_{12})_x\\
&\quad-\tfrac{1}{4}f_1[D_x^3 f_0\cdot f_{12}+3(D_x f_0\cdot f_{12})_{xx}]\\
&\overset{(4)}{=}-f_1(D_t f_0\cdot f_{12}+\tfrac{1}{4}D_x^3 f_0\cdot f_{12})-3kf_{1_{xx}}D_x f_1\cdot f_2\\
&\quad+3kf_{1_x}(D_x f_1\cdot f_2)_x-\frac{3k}{4}f_1(D_x f_1\cdot f_2)_{xx}\\
&=-f_1\left(D_t f_0\cdot f_{12}+\tfrac{1}{4}D_x^3 f_0\cdot f_{12}+\frac{3k}{4}D_x^3 f_1\cdot f_2\right)\\
&\overset{(7')}{=}0
\end{aligned}$$

which implies that

$$Q_2=(D_t+D_x^3)f_1\cdot f_{12}=0.$$

Similarly, we can show that $Q^2=(D_t+D_x^3)f_2\cdot f_{12}=0$. Therefore, f_{12} is a new solution of (2).

3. Particular solutions of the Ito equation

In this section we are going to derive some exact solutions of the Ito equation from BT (3′) and the nonlinear superposition formula (4). We seek particular solutions of the Ito equation according to the following steps. First, choose a given solution f_0 of (2). Secondly, from BT (3′) we find out f_1 and f_2 such that $f_0 \xrightarrow{\mu_i} f_i$ $(i=1,2)$ and, further, get a particular solution $\tilde{f}_{12}$ from (4). Then a general solution of (4) is $f_{12}=c(t)f_0+\tilde{f}_{12}$ (where $c(t)$ is an arbitrary function of t). Finally we substitute f_{12} into (5) and (7). If $c(t)$ can be determined such that $c_1(t)=c_2(t)=0$, the corresponding f_{12} is a new solution of the Ito equation (2). In what follows, we give four illustrative examples.

(i) Choose $f_0=1$. It is easily verified that

$$\begin{tikzcd}
 & 1+e^{\eta_1} \arrow[dr, "-k_2^3"] & \\
1 \arrow[ur, "-k_1^3"] \arrow[dr, "-k_2^3"'] & & \dfrac{k_1^3-k_2^3}{k_1^3+k_2^3}-e^{\eta_1}+e^{\eta_2}-\dfrac{k_1-k_2}{k_1+k_2}e^{\eta_1+\eta_2}. \\
 & 1+e^{\eta_2} \arrow[ur, "-k_1^3"'] &
\end{tikzcd}$$

Therefore $f_{12}=(k_1^3-k_2^3)/(k_1^3+k_2^3)-e^{\eta_1}+e^{\eta_2}-(k_1-k_2)/(k_1+k_2)\,e^{\eta_1+\eta_2}$ is a solution of the Ito equation (2), where $\eta_i=-k_ix+k_i^3t+\eta_i^0$ and k_i, η_i^0 are constants $(i=1,2)$.

(ii) It is easily verified that

$$\begin{tikzcd}
 & 1+x^2 \arrow[dr, "-k^3"] & \\
1 \arrow[ur, "0"] \arrow[dr, "-k^3"'] & & -1-x^2+\left(\dfrac{4}{k}x+1+x^2+\dfrac{4}{k^2}\right)e^{\eta}. \\
 & 1+e^{\eta} \arrow[ur, "0"'] &
\end{tikzcd}$$

Therefore $f_{12}=-1-x^2+[(4/k)x+1+x^2+4/k^2]\,e^{\eta}$ is a solution of the Ito equation (2), where $\eta=-kx+k^3t+\eta^0$ and k, η^0 are constants.

(iii) It is easily verified that

$$\begin{tikzcd}
 & t-\frac{1}{6}x^3 \arrow[dr, "-k^3"] & \\
1 \arrow[ur, "0"] \arrow[dr, "-k^3"'] & & -\dfrac{2}{k^3}-t+\frac{1}{6}x^3+\left(t-\frac{1}{6}x^3-\dfrac{1}{k}x^2+\dfrac{2}{k^2}x-\dfrac{2}{k^3}\right)e^{\eta}. \\
 & 1+e^{\eta} \arrow[ur, "0"'] &
\end{tikzcd}$$

Therefore $f_{12}=-2/k^3-t+\frac{1}{6}x^3+[t-\frac{1}{6}x^3-(1/k)x^2+(2/k^2)x-2/k^3]\,e^{\eta}$ is a solution of the Ito equation (2), where $\eta=-kx+k^3t+\eta^0$ and k, η^0 are constants.

(iv) It is easily verified that

$$\begin{tikzcd}
 & t-\frac{1}{6}x^3 \arrow[dr, "0"] & & \\
1 \arrow[ur, "0"] \arrow[r, "0"] \arrow[dr, "-k^3"'] & x \arrow[r, "0"] \arrow[dr, "-k^3"] & tx+\frac{1}{12}x^4 \arrow[r, "-k^3"] & \dfrac{2}{k^3}x+tx+\frac{1}{12}x^4 + \left(\dfrac{2}{k}t+xt+\frac{1}{12}x^4+\dfrac{2}{3k}x^3+\dfrac{2}{k^2}x^2+\dfrac{2}{k^3}x\right)e^{\eta}. \\
 & 1+e^{\eta} \arrow[r, "0"'] & -x+\left(\dfrac{2}{k}+x\right)e^{\eta} \arrow[ur, "0"'] &
\end{tikzcd}$$

Therefore $tx+\frac{1}{12}x^4$, $-x+(2/k)\,e^{\eta}+x\,e^{\eta}$ and $(2/k^3)x+tx+\frac{1}{12}x^4+[(2/k)t+xt+\frac{1}{12}x^4+(2/3k)x^3+(2/k^2)x^2+(2/k^3)x]\,e^{\eta}$ are solutions of the Ito equation(2), where $\eta=-kx+k^3t+\eta^0$ and k, η^0 are constants.

4. Nonlinear superposition formula of a model equation for shallow water waves

The so-called model equation for shallow water waves is [14, 18]

$$D_x(D_t-D_tD_x^2+D_x)f\cdot f=0. \tag{8}$$

Hirota has given a two-parameter BT for (8) [14]. We presented the following three-parameter BT for (8) in [16]

$$\begin{aligned}&[D_x^3+\gamma D_x^2+(\tfrac{1}{3}\gamma^2-1)D_x+\lambda]f\cdot f'=0\\&[3D_xD_t+\mu D_x+\gamma D_t+(\tfrac{1}{3}\gamma\mu-1)]f\cdot f'=0\end{aligned} \tag{9}$$

where λ, μ and γ are arbitrary constants. It is noticed that (8) is different from another model equation for shallow water waves [18, 14]:

$$D_x[D_x(D_t-\tfrac{2}{3}D_tD_x^2+D_x)f\cdot f]\cdot f^2-\tfrac{1}{3}D_t(D_x^4f\cdot f)\cdot f^2=0. \tag{10}$$

In [8] we have presented a nonlinear superposition formula of (10). In this section, we shall also present a nonlinear superposition formula of (8). For the sake of convenience in calculation, we set $\mu=\gamma=0$ in BT (9). In this case, (9) becomes

$$\begin{aligned}&(D_x^3-D_x+\lambda)f\cdot f'=0\\&(3D_xD_t-1)f\cdot f'=0.\end{aligned} \tag{9$'$}$$

In what follows, let f_0 be a solution of (8), $f_0\neq 0$. Suppose that f_i $(i=1,2)$ is a solution of (8) which is related by f_0 under BT (9′) with λ_i, i.e. $f_0\xrightarrow{\lambda_i}f_i$ $(i=1,2)$, and that f_{12} is defined by

$$D_xf_0\cdot f_{12}=kD_xf_1\cdot f_2 \qquad \text{(where } k \text{ is a non-zero constant).} \tag{11}$$

Thus, similar to the calculations in section 2, we can deduce respectively that

$$D_tf_1\cdot f_2+\frac{1}{k}D_tf_0\cdot f_{12}=c_1(t)f_0^2 \tag{12}$$

$$-D_xf_1\cdot f_2+\tfrac{1}{4}D_x^3f_1\cdot f_2+(\lambda_2-\lambda_1)f_1f_2+\frac{3}{4k}D_x^3f_0\cdot f_{12}=0 \tag{13}$$

$$-\frac{1}{k}D_xf_0\cdot f_{12}+\tfrac{3}{4}D_x^3f_1\cdot f_2+\frac{1}{4k}D_x^3f_0\cdot f_{12}+\frac{1}{k}(\lambda_1+\lambda_2)f_0f_{12}=c_2(t)f_0^2 \tag{14}$$

from

$$\begin{aligned}&[(3D_xD_t-1)f_0\cdot f_1]f_2-[(3D_xD_t-1)f_0\cdot f_2]f_1=0\\&[(D_x^3-D_x+\lambda_1)f_0\cdot f_1]f_2-[(D_x^3-D_x+\lambda_2)f_0\cdot f_2]f_1=0\\&[(D_x^3-D_x+\lambda_1)f_0\cdot f_1]_xf_2-[(D_x^3-D_x+\lambda_2)f_0\cdot f_2]_xf_1=0\end{aligned}$$

where $c_1(t)$ and $c_2(t)$ in (12) and (14) are some functions of t. Furthermore, if we assume that there exists an f_{12} determined by (11) such that $c_1(t)=c_2(t)=0$, we can show that the corresponding f_{12} is a new solution of (8) which is related to f_1 and f_2

under BT (9′). Similarly, we can also obtain some particular solutions of (8) by the use of the above result.

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Appendix

The following bilinear operator identities hold for arbitrary functions a, b, c and d:

$$(D_xD_ta\cdot b)c-(D_xD_ta\cdot c)b$$
$$=-a_xD_tb\cdot c-a_tD_xb\cdot c+\tfrac{1}{2}a[(D_xb\cdot c)_t+(D_tb\cdot c)_x] \tag{A1}$$

$$k_1(D_xa\cdot b)c-k_2(D_xa\cdot c)b$$
$$=(k_1-k_2)a_xbc-\tfrac{1}{2}a[(k_1+k_2)D_xb\cdot c+(k_1-k_2)(bc)_x] \tag{A2}$$

$$(D_ta\cdot b)c-(D_ta\cdot c)b=-aD_tb\cdot c \tag{A3}$$

$$(D_x^3a\cdot b)c-(D_x^3a\cdot c)b$$
$$=-3a_{xx}D_xb\cdot c+3a_x(D_xb\cdot c)_x-\tfrac{1}{4}a[D_x^3b\cdot c+3(D_xb\cdot c)_{xx}] \tag{A4}$$

$$(D_ta\cdot b)_xc-(D_ta\cdot c)_xb$$
$$=a_tD_xb\cdot c-a_xD_tb\cdot c-\tfrac{1}{2}a[(D_xb\cdot c)_t+(D_tb\cdot c)_x] \tag{A5}$$

$$(D_x^3a\cdot b)_xc-(D_x^3a\cdot c)_xb$$
$$=-2a_{xxx}D_xb\cdot c+\tfrac{1}{2}a_x[D_x^3b\cdot c+3(D_xb\cdot c)_{xx}]$$
$$-\tfrac{1}{2}a[(D_x^3b\cdot c)_x+(D_xb\cdot c)_{xxx}]. \tag{A6}$$

References

[1] Tu G Z 1979 *Appl. Math. Comput. Math.* **1** 21–43
[2] Tu G Z 1982 *Lett. Math. Phys.* **6** 63–71
[3] Hirota R and Satsuma J 1978 *J. Phys. Soc. Japan* **45** 1741–50
[4] Nakamura A 1979 *J. Phys. Soc. Japan* **47** 1335–40
[5] Nakamura A 1980 *J. Phys. Soc. Japan* **49** 2380–6
[6] Nakamura A 1981 *J. Math. Phys.* **22** 1608–13
[7] Nakamura A 1981 *J. Math. Phys.* **22** 2456–62
[8] Li Y and Hu X B 1986 *Sci. Expl.* **6**(4) 18–26
[9] Hu X B and Li Y 1988 *Acta Math. Appl. Sinica* **4**(1) 46–54
[10] Hu X B and Li Y 1989 *J. Tongji Univ.* **17**(1) 105–11
[11] Hu X B and Li Y 1991 *Acta Math. Scientia* **10**(2)
[12] Hu X B and Li Y 1989 Bilinearization of KdV, MKdV and classical Boussinesq hierarchies *Preprint* Computing Center of Academia Sinica
[13] Hu X B, Li Y and Liu Q M 1989 Nonlinear superposition formula of Boussinesq hierarchy *Preprint* Computing Center of Academia Sinica

[14] Bullough R K and Caudrey P J (eds) 1980 *Solitons, Topics in Current Physics* vol 17 (Berlin: Springer) pp 157–76
[15] Ito M 1980 *J. Phys. Soc. Japan* **49** 771–8
[16] Li Y and Hu X B 1989 *J. Tongji Univ.* **17**(3) 377–84
[17] Jimbo M and Miwa T 1983 *Publ. RIMS, Kyoto Univ.* **19** 943–1001
[18] Hirota R and Satsuma J 1976 *J. Phys. Soc. Japan* **40** 611–12